Logic and truth: Some logics without theorems

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Two types of logical consequence are compared: one, with respect to matrix and designated elements and the other with respect to ordering in a suitable algebraic structure. Particular emphasis is laid on algebraic structures in which there is no top-element relative to the ordering. The significance of this special condition is discussed. Sequent calculi for a number of such structures are developed. As a consequence it is re-established that the notion of truth as such, not to speak of tautologies, is inessential in order to define validity of an argument.

Keywords: logical truth, logical consequence, lattice

1. Introduction: Logical consequences vis a vis truth

The notion of truth serves various explanatory purposes. One of these is found in the attempted explanations of validity of deductive arguments. An argument is valid if and only if the conclusion of an argument is true whenever all the premises of the argument are also true. The context, here, is taken as classical and two-valued. When consequence is understood as a relation preserving truth, the notion of truth is taken to be that which relates semantics to states of affairs. Set theoretic models of well-formed formulae of 1^{st} order languages are the formal representation of states of affairs. Sometimes formulae (propositions) are modeled as sets of possible worlds, logical connectives by set theoretic operations and logical consequence by set theoretic relation. Though apparently there is no mention of 'truth' in this approach, the underlying intention is that a formula be associated with the set of those worlds (states of affairs) where the formula is true. Long back in 1920's many-valuedness (i.e., allowing sentences to have values other than true and false) was introduced and gradually accepted within the discourse of logic and

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philosophy. In many-valued context the notion of consequence has been defined in terms of some designated values equivalent in some sense to the value 'true' of two-valued contexts. In most of the significant many-valued logics, however, the designated set is taken to be a singleton set, i.e., simply the value 'true' is designated as before. If the notion of truth is to be of any use for explanation of the notion of validity of arguments, it has to be conceived in a way that accommodates behavior of truth with respect to the identifiable logical operations like conjunction, disjunction, negation, particularization, generalization etc. These kinds of behavior of truth get reflected when truth is studied in an algebraic setting on sets of so-called "truth values". These get reflected well when properties of truth relevant to logic are characterized mathematically. That these mathematically identifiable properties of truth are not out of tune with most other functions that this notion performs can be seen in the analysis of the notion of truth proposed by Tarski (although in the context of two-valuedness and of formalized languages only). It should also be noted that the notion of logical truth or tautologihood plays a fundamental role in logic-systems that depend upon truth-semantics, in particular, in framing axiomatic theories. Up to a point of time in history, logic was engaged in studies of tautologies or universally valid sentences, giving rise to criticisms of spending energy in dealing with "contentless" assertions. Subsequently, in axiomatic theories, usually two kinds of axioms are considered, logical and proper—the first kind being sentences that are true always, in all situations and the second kind being sentences that are true in specific situations or models. It is only recent in the long history of logic that logical consequence rather than logical truth, has taken the centre-stage. Logical consequence is usually defined in two different ways. A well-formed formula (wff) A is a logical consequence of a set of wffs Γ (written as $\Gamma \vDash_1 A$ in this paper) if and only if in any situation (represented by a valuation) if every member of Γ receive values in the set D of designated elements in a suitable matrix, A also receives a value in D. Another way of defining the consequence is that for every valuation v in any algebra of an appropriate class, $v(\Gamma)$ is less or equal to $v(A)$, where $v(\Gamma)$ means some kind of composition of the values under v of each component of Γ. The second kind of consequence shall be denoted by $\Gamma \vDash_2 A$ in the sequel. These two approaches are not equivalent. When the two relations coincide, it becomes necessary that there shall be only one designated element in the value-set and it shall be the topmost element of the set relative to the ordering. This point will be further clarified in the following section. But there are various systems with more than one designated values. Even if we consider only one designated value, it is not guaranteed that the system contains tautologies or theorems. In fact, Leo Simons (1974, 1978) proposed a system C which is

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equivalent to Copi's system (Copi 1998). System *C* is peculiar in that in it formulae which are by classical characterizations tautologies or theorems may not be a *C*-theorem. In this sense, the system *C* contains no tautologies. The logic-system *B* developed by Belnap and Font (Belnap 1977, Font 1997) with semantics given in the 4-valued Belnap lattice and Kleene 3-valued logic *K* with strong negation do not have theorems at all. So, instead of beginning with a notion of truth one can try to understand validity by focusing only upon the mathematically characterizable properties of the values assignable to the conclusion in relation to those assignable to the premises. The usual algebraic structures in which semantics of various logic-systems are defined admit of bounds viz., the top and the least elements and there are already existing logical systems with semantic consequence relations defined in terms of ordering in such structures. On the other hand, it is quite possible and natural also, to think of algebraic structures that do not essentially require a top element for their basic defining properties. So a natural query is: what changes in the logic systems have to be made in order to have their semantics, i.e., consequence relation fully determined by these structures without top elements or making any reference to the top element even though it exists. Such logics obviously do not have tautologies and as completeness can be proved, they do not have any theorems either.

One might argue that in the use of order relations of an algebraic structure in the semantics, the role of truth is not abandoned but merely shifted to a comparative notion viz., 'more true than'. And indeed, it is definitely so if 'degree of truth' is accepted and even non-comparability of truth-degrees is also admitted. Moreover when the highest and lowest elements of the algebra are absent, one should be ready to give up the notions of absolute or full true and absolute or full false altogether. Neither of the above two ideas are readily acceptable by the classicists. Yet, more importantly, an attempt to give semantics to logical consequence by ordering relation opens up other possibilities too. Something (other than truth) is being preserved by the consequence—that may be meaning, information or acceptability or something else.

In this paper we shall give further instances of logics in which one talks of validity of an argument (or soundness of the consequence relation) without any reference to the set of designated elements or 'truth'. However, the purpose of this paper is not limited to that only but to emphasize upon the fact that a valid consequence may completely ignore the notion of truth, be it absolute or comparative. The work points at the existence of a property of sentences different from truth which facilitates consequence, a property which need not have an extreme value or bounds but has to be only algebraically characterizable.

2. Consequence relation without truth

Formally, an argument is defined as a logical consequence or semantic consequence relation denoted by $\vDash$. Let $\Gamma = \{A_1, A_2, \ldots, A_n\}$ be a set of wffs, A a single wff and v a valuation function from the set of wffs to a suitable algebraic structure. We shall define $\vDash$ in two ways and for clarity shall denote them by $\vDash_1$ and $\vDash_2$. $\Gamma \vDash_1 A$ if and only if for all v, $v(A_i) \in D$ for all $A_i \in \Gamma$ implies $v(A) \in D$, where D is a set of designated elements of the algebra.

Alternatively, let $v(\Gamma)$ denote some kind of composition of the values of $v(A_1), v(A_2), \ldots, v(A_n)$, the composition being available in the algebraic structure usually the composition for computing the conjunction. Another way of defining the consequence relation is: $\Gamma \vDash_2 A$ if and only if for all v, $v(\Gamma) \leq v(A)$, where $\leq$ is a suitable ordering in the structure.

In the standard 2-valued (values being 0 and 1, say) classical logic with $D = \{1\}$ and natural ordering $\leq$, the two notions are equivalent. Here $v(\Gamma)$ is taken as $v(A_1) \wedge v(A_2) \wedge \cdots \wedge v(A_n)$, $\wedge$ being the composition for classical conjunction. Let $\Gamma \vDash_1 A$. Now, $v(\Gamma)$ is either 0 or 1. If $v(\Gamma)$ is 0, then $v(\Gamma) \leq v(B)$ always. If $v(\Gamma)$ is 1, then $v(A_i) = 1$ for all $A_i \in \Gamma$. So, by the definition of $\vDash_1$, $v(B) = 1$. Then $v(\Gamma) \leq v(B)$. Thus $\Gamma \vDash_2 A$. Conversely, let $\Gamma \vDash_2 A$ and let $v(A_i) = 1$ for all $A_i \in \Gamma$. Then $v(\Gamma) = 1$. So, by the definition of $\vDash_2$, $v(B) = 1$. That is $\Gamma \vDash_1 A$.

The above equivalence depends on the fact that

1. $v(\Gamma) = 1$ if and only if $v(A_i) = 1$ for all $A_i \in \Gamma$, i.e., the composition operator $\wedge$ is such that $x \wedge y = 1$ if and only if $x = 1$ and $y = 1$.

2. 1 is the greatest element relative to the ordering.

3. 1 is the designated and the only designated element.

It is to be noted that 1, 2 and 3 are crucial for establishing the above equivalence.

Let us ignore the ordering and consider the algebraic structure only with any subset D as designated and define $\vDash_1$. It should be noted that the set D needs to be endowed with certain restrictions depending upon the particular logic-system. However, the following properties of $\vDash_1$ are derivable for any subset D. The relation satisfies the conditions, the basic requirement of a logical consequence relation due to Gentzen (1969).

(i) $A \in \Gamma$ implies $\Gamma \vDash_1 A$ (overlap)

(ii) $\Gamma \subseteq \Delta, \Gamma \vDash_1 A$ implies $\Delta \vDash_1 A$ (weakening or monotonicity)

(iii) If for all $B \in \Delta, \Gamma \vDash_1 B$ and $\Delta \vDash_1 A$ then $\Gamma \vDash_1 A$ (cut)

So, weakening comes automatically if the logic is defined by designated set. Let $v(B) = 1$ for all $B \in \Delta$. Then $v(B) = 1$ for all $B \in \Gamma$. So, $v(A) = 1$. Hence, if a logic intends to avoid weakening (e.g., non-monotonic logic), it can not be defined in terms of a designated element. From the angle of sequent calculus, the condition (ii) is left weakening. It may be mentioned that if logic is defined in terms of a designated set, then the right weakening also holds.

Even in this approach, there may be a logic without there being tautologies, although elements of the designated set may be considered as the counterpart of 'True' of 2-valued standard semantics.

Let us consider the following truth tables of Kleene's system.

A	$\sim A$
1	0
1/2	1/2
0	1

$\to$	1	1/2	0
1	1	1/2	0
1/2	1	1/2	1/2
0	1	1	1

$\wedge$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	0	0	0

$\vee$	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

So the matrix is $\langle\{1, 1/2, 0\}, \sim, \to, \wedge, \vee, \{1\}\rangle$. Here if any valuation assigns a value 1/2 to every propositional variable present in a wff, the valuation assigns 1/2 to that wff as a whole. So, there are no tautologies although one may define the logical consequence $\vDash_1$ relative to this matrix. It may be mentioned here that Leo Simons also considered the same matrix in the discussion of his system C (Simons 1974, 1978). Thus the standard matrix method for defining consequence $\vDash_1$ may not find a formula A in some logic-system so that $\vDash_1 A$ holds, or in other words, A is a tautology. Some comparisons between the two consequences are shown below.

Let $\vDash_1$ be defined in terms of designated value set D. For any two formulae A and B, we define $A \leq B$ if and only if for all $v, v(A) \in D$ implies $v(B) \in D$. This relation is reflexive and transitive. We now define $A \equiv B$ if and only if $A \leq B$ and $B \leq A$. This is an equivalence relation. Hence the set of wff are clustered in classes of mutually equivalent elements. Now, we lift the order relation $\leq$ among equivalence classes by $[A] \leq [B]$ if and only if $A \leq B$. This definition is unambiguous. The operations of the language are usually lifted too as follows: $\sim[A] := [\sim A]$, $[A] \wedge [B] := [A \wedge B]$, $[A] \vee [B] := [A \vee B]$, $[A] \to [B] := [A \to B]$. Because of the standard conditions imposed on the use of connectives the operation ≡ turns out to be a congruence and hence there occurs no ambiguity in the definition.

Let A be called a tautology if and only if $v(A) \in D$ for all valuations v in the original matrix relative to which $\vDash_1$ is defined. It is easy to check that tautologies form an equivalence class. If this class is non-empty, then it has to be the greatest element among the clusters. However, this class may be empty, i.e., there may be no tautologies of the logic although there is the notion of logical consequence in it. Hence the clusters of wffs are partially ordered with a topmost cluster (if there is any tautology at all) which is the cluster of tautologies with respect to the designated set. We can think of there being the ordering without there being tautologies. An algebraic structure is thus formed of the equivalence class (the quotient algebra) with or without a topmost element.

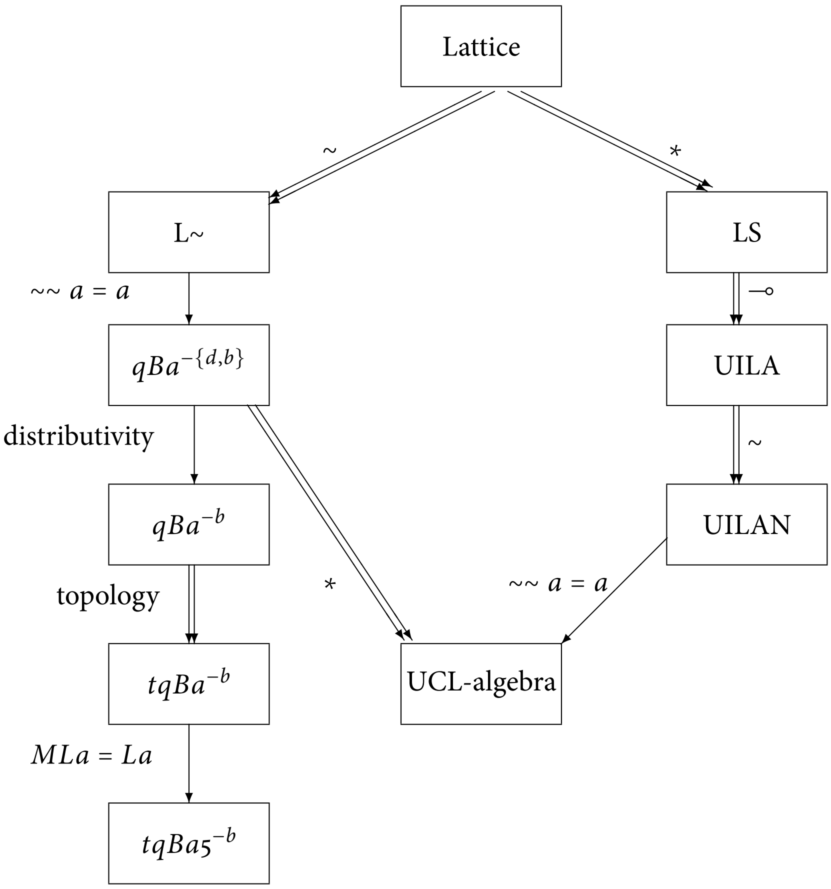
We now make the following comparison between the two logical consequence $\vDash_1$ defined above and $\vDash_2$ to be defined below. Let $A \vDash_2 B$ hold if and only if for all valuations V in the quotient algebra formed above, $V(A) \leq V(B)$. Since canonical mapping viz., $A \to [A]$ turns out to be a valuation, we have that $[A] \leq [B]$ when $A \vDash_2 B$ holds. Hence, according to construction, $A \vDash_1 B$ holds. So, $\vDash_2$ defined in this way is more restrictive than $\vDash_1$. On the other hand, in general consequence of the second type $\vDash_2$ may include both monotonic and non-monotonic consequences.

However, we concentrate on $\vDash_2$. In the next section, we discuss logics for which bounds, or more specifically, a topmost element in the value set is not a necessity, but yet an inference machinery may be meaningfully defined relative to an ordering in it. The general framework of sequent calculus shall be adopted in the following section with the usual syntax-semantic divide where the semantics is a generalization of $\vDash_2$.

3. Logic for algebras not necessarily bounded

The following diagram shows a hierarchy of algebraic systems starting with the simplest one and branching into two directions. In one direction, a ~-operator is taken and gradually further axioms are added. In the other direction, a binary operation $(*)$ is added and other systems with further axioms are generated. The meeting point of the two branches is shown. It should be noted that quite a few of these algebras with essentially top and least elements are models of some significant logic-systems such as multiplicative additive linear logic (UCL-algebra), topological quasi-Boolean logic ($tqBa5^{-b}$), a close associate of pre-rough logic etc. (Sen and Chakraborty 2002, Girard 1987, Troelstra 1992). While there are theorems in the existing logics because of the existence of topmost elements in the algebraic structures in the systems presented here there shall be none. That means the following logical systems possess valid consequence relations without any essential reference to tautologihood or even a comparative notion of truth.

In the diagram, $X \longrightarrow Y$ means X with some additional algebraic axiom(s) gives Y and $X \Longrightarrow Y$ means X with some additional structure(s) gives Y and rules of the logic corresponding to the algebra Y need to be changed.



Let us begin with the least structure on the value set as a lattice (in general, without bounds). The premise set Γ is taken to be finite. In fact, we take multisets to allow repeated occurrence of an element such as $\{A_1, A_1, A_2, \ldots, A_n\}$.

Definition 3.1 A lattice $\langle X, \wedge, \vee \rangle$ is a structure where X is a non-empty set and $\wedge, \vee$ are two binary operations on X satisfying for all $a, b, c \in X$, the conditions

(i) $a \wedge b = b \wedge a,\ a \vee b = b \vee a$ (commutative property)

(ii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c,$
$a \vee (b \vee c) = (a \vee b) \vee c$ (associative property)

(iii) $a \vee (a \wedge b) = a = a \wedge (a \vee b)$ (absorption property)

The relation $\leq$, defined by $a \leq b$ if and only if $a \wedge b = a$ or equivalently $a \vee b = b$ is a partial order relation, $a \wedge b$ and $a \vee b$ being the greatest lower bound and the least upper bound of a, b respectively. Now, we introduce a sequent calculus Lattice Logic (**LL**) for lattice. Language of the logic of lattices, consists of p_i's, the propositional variables (atomic sentences) and $\wedge, \vee$ the binary logical connectives conjunction and disjunction (following usual abuse of notations). Formulae are formed as usual and they are denoted by A, B etc. A sequent is of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulae. A sequent may be treated as a generalization of the notion of logical consequence since on the right hand side now there is a finite multiset instead of a single formula. The symbol $\Rightarrow$ is used to indicate the distinction.

Now, we shall state the axiom and rules of the logic.

$$A \Rightarrow A \;\; Ax \qquad\qquad \frac{\Gamma \Rightarrow A \quad \Gamma', A \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta}\, Cut$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\, LW \qquad\qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}\, RW$$

$$\frac{\Gamma', A, A \Rightarrow \Delta}{\Gamma', A \Rightarrow \Delta}\, LC \qquad\qquad \frac{\Gamma \Rightarrow A, A, \Delta'}{\Gamma \Rightarrow A, \Delta'}\, RC$$

$$\frac{A \Rightarrow \Delta \quad B \Rightarrow \Delta''}{A \vee B \Rightarrow \Delta, \Delta''}\, L\vee \qquad\qquad \frac{\Gamma \Rightarrow A, B, \Delta'}{\Gamma, \Rightarrow A \vee B, \Delta'}\, R\vee$$

$$\frac{\Gamma', A, B \Rightarrow \Delta}{\Gamma', A \wedge B \Rightarrow \Delta}\, L\wedge \qquad\qquad \frac{\Gamma \Rightarrow A \quad \Gamma'' \Rightarrow B}{\Gamma, \Gamma'' \Rightarrow A \wedge B}\, R\wedge$$

where $\Gamma, \Gamma'', \Delta, \Delta'' \neq \varnothing$.

The deviations from the standard $L\vee$ and $R\wedge$ rules of classical propositional logic may be noted. By convention, to denote the multisets we use Γ's as the left hand side and Δ's as the right of the $\Rightarrow$ in a sequent.

Definition 3.2 A lattice-model is a lattice $\langle X, \wedge, \vee \rangle$ with a valuation $\|\ \ \|$ assigning a value $\|p\| \in X$ to each atomic sentence p in the language.

$\|\ \ \|$ is extended to arbitrary formulae by $\|A \wedge B\| = \|A\| \wedge \|B\|$ and $\|A \vee B\| = \|A\| \vee \|B\|$. A sequent $\Gamma \Rightarrow \Delta$ is said to be valid in a lattice-model $(\langle X, \wedge, \vee \rangle, \|\ \ \|)$ if and only if $\|A_1\| \wedge \|A_2\| \wedge \ldots \wedge \|A_m\| \leq \|B_1\| \vee \|B_2\| \vee \ldots \vee \|B_n\|$ which we shall write as $\|\Gamma\| \leq \|\Delta\|$, where Γ is $A_1, A_2, \ldots, A_m$ and Δ is $B_1, B_2, \ldots, B_n$.

Theorem 3.1 (Soundness) If $\Gamma \Rightarrow \Delta$ is derivable in Lattice Logic, then $\Gamma \Rightarrow \Delta$ is valid in every lattice-model.

To prove the theorem mathematical induction on the depth of derivation is used. For this, we shall check that the axiom is valid and all the rules preserve validity.

Theorem 3.2 (Completeness) If $\Gamma \Rightarrow \Delta$ is valid in every lattice-model, then $\Gamma \Rightarrow \Delta$ is derivable in Lattice Logic.

To prove this theorem, we first construct the Lindenbaum algebra. A relation ρ is defined on the set **F** of wffs by $A\rho B$ if and only if $A \Rightarrow B$ and $B \Rightarrow A$ are derivable. It is easy to check that ρ is an equivalence relation on **F**. The quotient algebra $\mathbf{F}/\rho$ is then formed in the usual way with the equivalence classes $[A]$ for each wff A. It can be shown that the compositions defined on the quotient algebra are independent of the choice of the representatives of the equivalence classes. Now, it has to be proved that the quotient algebra, i.e., the Lindenbaum algebra is a lattice. This proves completeness, since if $\Gamma \Rightarrow \Delta$ is valid in every lattice-model, i.e., if $\|\Gamma\| \leq \|\Delta\|$ holds for every member of the class of models and every valuation $\|\ \ \|$, it holds in the Lindenbaum algebra with the canonical valuation, i.e., when A is mapped to its equivalence class $[A]$. Thus $[\Gamma] \leq [\Delta]$ which implies $\Gamma \Rightarrow \Delta$ is derivable in Lattice Logic. A sketch of the proof of this claim is shown below.

Let $\Gamma = A_1, A_2, ..., A_m$ and $\Delta = B_1, B_2, ..., B_n$. Then $[\Gamma] \leq [\Delta]$ means $[A_1] \wedge [A_2] \wedge ... \wedge [A_m] \leq [B_1] \vee [B_2] \vee ... \vee [B_n]$. So, $[A_1 \wedge A_2 \wedge ... \wedge A_m] \leq [B_1 \vee B_2 \vee ... \vee B_n]$. For simplicity, we write A for $A_1 \wedge A_2 \wedge ... \wedge A_m$ and B for $B_1 \vee B_2 \vee ... \vee B_n$. Then $[A] \leq [B]$, i.e., $[A] \vee [B] = [B]$. So, $[A \vee B] = [B]$. Hence $A \vee B \Rightarrow B$ and $B \Rightarrow A \vee B$ are derivable. In lattice logic, $A \Rightarrow A \vee B$ is derivable. Hence, by Cut, $A \Rightarrow B$ is derivable, i.e., $A_1 \wedge A_2 \wedge ... \wedge A_m \Rightarrow B_1 \vee B_2 \vee ... \vee B_n$ is derivable. Using Cut twice, $A_1, A_2, ..., A_m \Rightarrow B_1, B_2, ..., B_n$ i.e., $\Gamma \Rightarrow \Delta$ is derivable.

Lattice with negation $(L\sim)$ is an algebra $\langle X, \leq, \wedge, \vee, \sim \rangle$ where $\langle X, \leq, \wedge, \vee \rangle$ is a lattice and $\sim$ is an unary operator satisfying the property, if $a \leq b$ then $\sim b \leq \sim a$ for all $a, b \in X$. In the corresponding logic $LL\sim$, we consider all the axioms and rules of LL and also take one extra rule

$$\frac{A \Rightarrow B}{\sim B \Rightarrow \sim A}\ (Rule\sim)^r$$

Here by $(Rule\sim)^r$ we mean the restricted rule of negation. $LL\sim$ is sound and complete with respect to the algebra $L\sim$.

The algebra $\boldsymbol{qBa^{-\{d,b\}}}$ is $L\sim$ and here $\sim$ satisfies $\sim\sim a = a$ for all a in the lattice. In this algebra $\sim(a \wedge b) = \sim a \vee \sim b$ and $\sim(a \vee b) = \sim a \wedge \sim b$. We use the symbol $qBa^{-\{d,b\}}$, as it is actually quasi-Boolean algebra without distributivity and boundedness axioms. Corresponding logic is $qBl^{-\{d,b\}}$ which is basically $LL\sim$ with two extra axioms

$$A \Rightarrow \sim\sim A \;\; Ax1 \qquad\qquad \sim\sim A \Rightarrow A \;\; Ax2$$

Due to Cut, one can consider only $Ax1$ and $Ax2$ as axioms and Ax $A \Rightarrow A$ need not be taken as axiom here. In this logic, the rule

$$\frac{\Gamma \Rightarrow \Delta}{\sim\Delta \Rightarrow \sim\Gamma}\,(Rule\sim)$$

follows for $\Gamma, \Delta \neq \emptyset$. Here $\sim\Gamma$ is $\sim A_1, \sim A_2, \ldots, \sim A_n$ for $\Gamma = A_1, A_2, \ldots, A_n$. This logic is sound and complete with respect to the algebra $qBa^{-\{d,b\}}$.

If we consider distributivity in the algebra $qBa^{-\{d,b\}}$, we get qBa^{-b}. It is actually quasi-Boolean algebra except boundedness. In the corresponding logic qBl^{-b} we take $Ax1$ and $Ax2$ of $qBl^{-d,b}$ as axioms. Also the rules $LW, RW, LC, RC, R\vee, L\wedge$ and $(Rule\sim)$ of $qBl^{-d,b}$ are taken here. But we replace the rules $Cut, L\vee$ and $R\wedge$ by

$$\frac{\Gamma \Rightarrow A, \Delta' \quad \Gamma', A \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,Cut'$$

$$\frac{\Gamma', A \Rightarrow \Delta \quad \Gamma_1, B \Rightarrow \Delta''}{\Gamma', \Gamma_1, A \vee B \Rightarrow \Delta, \Delta''}\,L\vee'$$

and

$$\frac{\Gamma \Rightarrow A, \Delta' \quad \Gamma'' \Rightarrow B, \Delta_1}{\Gamma, \Gamma'' \Rightarrow A \wedge \Delta', \Delta_1}\,R\wedge'$$

respectively (with the same restriction $\Gamma, \Gamma'', \Delta, \Delta'' \neq \emptyset$). In this sequent calculus, we have to change the rules to incorporate distributivity. Soundness and completeness also hold here.

Topological $\boldsymbol{qBa^{-b}}$, i.e., $tqBa^{-b}$ is qBa^{-b} with an unary operator L satisfying $La \leq a, L(a \wedge b) = La \wedge Lb$ and $LLa = La$ for all a, b in the underlying set. $\boldsymbol{tqBa5^{-b}}$ is $tqBa^{-b}$ with $MLa = La$ for all a in the set, where $M \equiv \sim L\sim$. Corresponding logics are $tqBl^{-b}$ which is qBl^{-b} with an unary connective l and the rules

$$\frac{\Gamma', A \Rightarrow \Delta}{\Gamma', lA \Rightarrow \Delta} Ll \qquad \frac{l\Gamma \Rightarrow A}{l\Gamma \Rightarrow lA} Rl$$

(for $\Gamma, \Delta, \neq \varnothing$).

Another logic is $tqBl5^{-b}$ which is $tqBl^{-b}$ with $Ax3, mlA \Rightarrow lA$, where $mA := {\sim}l{\sim}A$. Both are sound and complete with respect to the corresponding algebras.

Lattice with commutative semigroup (LS) is an algebra $\langle X, \leq, \wedge, \vee, *\rangle$ where $\langle X, \leq, \wedge, \vee\rangle$ is a lattice and $\langle X, *\rangle$ is a commutative semigroup. Also if $a \leq b$ then $a * c \leq b * c$ for all $a, b, c \in X$. The axiom and rules of the corresponding logic $L*$ are as follows:

$$A \Rightarrow A \;\; Ax \qquad \frac{\Gamma \Rightarrow A \quad \Gamma', A \Rightarrow B}{\Gamma, \Gamma' \Rightarrow B} Cut$$

$$\frac{A \Rightarrow C \quad B \Rightarrow C}{A \vee B \Rightarrow C} L\vee \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B} \; \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B} R\vee$$

$$\frac{\Gamma', A \Rightarrow C}{\Gamma', A \wedge B \Rightarrow C} \; \frac{\Gamma', B \Rightarrow C}{\Gamma', A \wedge B \Rightarrow C} L\wedge \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} R\wedge$$

$$\frac{\Gamma', A, B \Rightarrow C}{\Gamma', A * B \Rightarrow C} L* \qquad \frac{\Gamma \Rightarrow A \quad \Gamma'' \Rightarrow B}{\Gamma, \Gamma'' \Rightarrow A * B} R*$$

where $\Gamma, \Gamma'' \neq \varnothing$. The deviations from the standard $L\vee$, $R\vee$, $L\wedge$ and $R\wedge$ rules of previous logic may be noted.

Definition 3.3 A LS-model is a LS $\langle X, \leq, \wedge, \vee, *\rangle$ with a valuation $\| \quad \|$ assigning a value $\|p\| \in X$ to each atomic sentence p in the language.

$\| \quad \|$ is extended to arbitrary formulae by $\|A \wedge B\| = \|A\| \wedge \|B\|$, $\|A \vee B\| = \|A\| \vee \|B\|$ and $\|A * B\| = \|A\| * \|B\|$.

A sequent $\Gamma \Rightarrow A$ is said to be valid in a LS-model $(\langle X, \leq, \wedge, \vee, *\rangle, \| \quad \|)$ if and only if $\|A_1\| * \|A_2\| * \ldots * \|A_m\| \leq \|A\|$ which we shall write as $\|\Gamma\| \leq \|A\|$, where Γ is $A_1, A_2, \ldots, A_m$. In the proof of completeness, $[\Gamma]$ is the $*$-operation of the class of the elements of Γ. It is easy to check that the logic $L*$ is sound and complete with respect to the algebra LS.

Intuitionistic linear algebra (ILA) is the algebra for intuitionistic linear logic (ILL). An $UILA$ is an algebra where $\langle X, \leq, \wedge, \vee, *, \multimap\rangle$ where $\langle X, \leq, \wedge, \vee, *\rangle$ is a LS and

(i) if $a \leq b$ then $c \multimap a \leq c \multimap b$ and $b \multimap c \leq a \multimap c$ for all $a, b, c \in X$.

(ii) $a * b \leq c$ if and only if $a \leq b \multimap c$ for all $a, b, c \in X$.

It may be noted that in $UILA$ the lattice is not bounded, in general and $\langle X, * \rangle$ is a semigroup, not a monoid. In the corresponding logic $UILL$, we consider all axiom and rules of $L*$ with two more rules for $\multimap$ (a new binary connective).

$$\frac{\Gamma \Rightarrow A \quad \Gamma', B \Rightarrow C}{\Gamma, \Gamma', A \multimap B \Rightarrow C} L \multimap \qquad\qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \multimap B} R \multimap$$

with the same restriction $\Gamma \neq \varnothing$.

UILAN is the algebraic structure $UILA$ with an unary operator negation ($\sim$) which satisfies the rule: if $a \leq b$ then $\sim b \leq \sim a$ for all a, b in the underlying set. The corresponding logic $UILLN$ is the same as the logic $UILL$ with one more rule $(Rule\sim)^r$ (mentioned earlier) for negation (a new unary connective).

Finally, the algebra ***UCL*-algebra** is *UILAN* with the restriction $\sim\sim a = a$ for all a in the set. *UCL*-algebra differs from classical linear algebra (the algebra for classical linear logic) in two points:

1) The lattice may be unbounded.

2) The binary composition $*$ satisfies semigroup properties, there may not exist any identity with respect to $*$.

The corresponding logic $UCLL$ is the same as $UILLN$, but we replace Ax, $A \Rightarrow A$, by two axioms $Ax1$ and $Ax2$ (mentioned earlier). Soundness and completeness of last three logics are also obtained.

4. Conclusion

We have established that top element or even a designated set of elements mimicking "true" of the two-valued logic is not required to form a logic. The familiar notion of truth which performs so many other functions and also performs the function of explaining validity is not of much importance here. On the other hand, what is philosophically important is that, by the latter approach attempts are being made to identify the bare essentials that have to be satisfied for validity to emerge in an argument. Such an attempt in understanding the notion of validity by by-passing the use of the notion of truth will only help in seeing the exact relation between logic and the standard notion of truth.

Besides, speaking technically, if the validity of an argument is ascertained in terms of the designated elements of a matrix then it can be verified by the second method with respect to a specific algebraic structure obtained by the Lindenbaum construction and a special valuation function, viz., the canonical valuation.

It is also interesting to note that Font says that "the lack of theorems of the logic *B* and *K* can be remedied, if we want, by adding them artificially; enlarge the language with propositional constants $\top$ (truth) or $\perp$ (falsum)." But in our opinion lack of theoremhood may not be looked upon as a defect of the logic that deserves to be "remedied". On the other hand this fact opens up new insight about the notion of validity of an argument by delinking it from the notion of truth.

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